Basic Dirac operator and transversal twistor operator

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Abstract. We study the transversal killing and twistor spinor on a foliated Riemannian manifold with a transverse spin structure. In particular, we prove the lower bound for the basic Dirac operator on the Riemannian foliation by using the transversal twistor operator.

1. Introduction

Twistor spinors were introduced by R. Penrose in General Relativity([18]). In [15], A. Lichnerowicz introduced the twistor operator on the spinors, which is a conformally invariant, and proved that the twistor spinors are zeros of the twistor operator. Further, it is remarkable that the twistor spinors correspond to parallel sections in a certain bundle (see [2] or [5]).

Let \((M, g_M, F)\) be a Riemannian manifold with a transverse spin foliation \(F\), a bundle-like metric \(g_M\). In [9], the author have introduced the transversal Killing spinor which is given by the solution of the equation

\[
\nabla_X \Psi + f\pi(X) \cdot \Psi = 0 \quad \text{for} \quad X \in TM,
\]

where \(\pi : TM \to Q\) is a projection (see (2.1)). It is well known [9] that any eigenvalue of the basic Dirac operator \(D_b\) satisfies the inequality

\[
\lambda^2 \geq \frac{q}{4(q-1)} \inf_M (\sigma^\nabla + |\kappa|^2)
\]

where \(q = \operatorname{codim} F\), \(\sigma^\nabla\) is a transversal scalar curvature and \(\kappa\) is the mean curvature form of \(F\). And in the limiting case, \(M\) admits a transversal Killing spinor.

In this paper, we introduce the transversal twistor spinors and investigate their properties on \((M, g_M, F)\). Moreover, we study the properties of the transversal Killing spinor which occurs in the limiting case. By using the Weitzenböck-type formula about the transversal twistor operator, we estimate the sharper estimation for the eigenvalue of the basic Dirac operator.

The paper is organized as follows. In Section 2, we review the known facts on the foliated Riemannian manifold. In Section 3, we study the transversal twistor
spinor, which satisfy the transversal twistor equation

\[ \nabla_X \Psi + \frac{1}{q} \pi(X) \cdot D'_{tr} \Psi = 0 \quad \text{for any } X \in TM. \]

Moreover, we prove that the basic W-twistor spinors correspond to parallel basic sections in \( E = S(F) \oplus S(F) \). In Section 4, we study the transversal Killing spinor. Many definitions in this paper are defined similarly to those on an ordinary manifold. In last section, we estimate the eigenvalue of the basic Dirac operator, which is sharper than (1.2).

Throughout this paper, we consider the bundle-like metric \( \tilde{g}_M \) for \((M, F)\) such that the mean curvature form \( \kappa \) is basic and harmonic. The existence of the bundle-like metric \( g_M \) for \((M, F)\) such that \( \kappa \) is basic, i.e., \( \kappa \in \Omega^1_B(F) \), is proved in [4]. In [16,17], for any bundle-like metric \( g_M \) with \( \kappa \in \Omega^1_B(F) \), it is proved that there exists another bundle-like metric \( \tilde{g}_M \) for which the mean curvature form \( \tilde{\kappa} \) is basic-harmonic.

2. Preliminaries and known facts

In this section, we review the basic properties of the Riemannian foliation, which are studied in [12,19]. Let \((M, g_M, F)\) be a \((p+q)\)-dimensional Riemannian manifold with a foliation \( F \) of codimension \( q \) and a bundle-like metric \( g_M \) with respect to \( F \). We recall the exact sequence

\begin{equation}
0 \to L \to TM \overset{\pi}{\to} Q \to 0
\end{equation}

determined by the tangent bundle \( L \) and the normal bundle \( Q = TM/L \) of \( F \). The assumption of \( g_M \) to be a bundle-like metric means that the induced metric \( g_Q \) on the normal bundle \( Q \cong L^\perp \) satisfies the holonomy invariance condition \( \tilde{\nabla} g_Q = 0 \), where \( \tilde{\nabla} \) is the Bott connection in \( Q \).

For a distinguished chart \( U \subset M \) the leaves of \( F \) in \( U \) are given as the fibers of a Riemannian submersion \( f : U \to V \subset N \) onto an open subset \( V \) of a model Riemannian manifold \( N \).

For overlapping charts \( U_\alpha \cap U_\beta \), the corresponding local transition functions \( \gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1} \) on \( N \) are isometries. Further, we denote by \( \nabla \) the canonical connection of the normal bundle \( Q \) of \( F \). It is defined by

\begin{equation}
\begin{cases}
\nabla_X s = \pi([X,Y_s]) & \text{for } X \in \Gamma L, \\
\nabla_X s = \pi(\nabla^{\perp} Y_s) & \text{for } X \in \Gamma L^\perp,
\end{cases}
\end{equation}

where \( s \in \Gamma Q \), and \( Y_s \in \Gamma L^\perp \) corresponding to \( s \) under the canonical isomorphism \( L^\perp \cong Q \). The connection \( \nabla \) is metric and torsion free. It corresponds to the Riemannian connection of the model space \( N \). The curvature \( R^{\nabla} \) of \( \nabla \) is defined by

\[ R^{\nabla}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla\{X,Y\} \quad \text{for } X, Y \in TM. \]
Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L([12])$, we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature $\sigma^\nabla$ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1, \ldots, q}$ is an orthonormal basic frame for $Q$. $\mathcal{F}$ is said to be (transversally) Einsteinian if the model space $N$ is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q} \sigma^\nabla \cdot \text{id}$$

with constant transversal scalar curvature $\sigma^\nabla$.

The second fundamental form of $\alpha$ of $\mathcal{F}$ is given by

$$\alpha(X, Y) = \pi(\nabla^M_X Y) \quad \text{for } X, Y \in \Gamma L.$$

It is trivial that $\alpha$ is $Q$-valued, bilinear and symmetric.

The mean curvature vector field of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \alpha(E_i, E_i),$$

where $\{E_i\}_{i=1, \ldots, p}$ is an orthonormal basis of $L$. The dual form $\kappa$, the mean curvature form for $L$, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \text{for } X \in \Gamma Q.$$

The foliation $\mathcal{F}$ is said to be minimal (or harmonic) if $\kappa = 0$.

Let $\Omega^*_B(\mathcal{F})$ be the space of all basic $r$-forms, i.e.,

$$\Omega^*_B(\mathcal{F}) = \{ \psi \in \Omega^r(M) \mid i(X)\phi = 0, \theta(X)\phi = 0, \text{ for } X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be isoparametric if $\kappa \in \Omega^1_B(\mathcal{F})$. We already know that $\kappa$ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([19]). Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega^*_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. Let $\delta_B$ the adjoint operator of $d_B$. Then it is well-known([1,9]) that

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B^\sharp),$$

where $\kappa_B^\sharp$ is the $g_Q$-dual vector field of the basic component $\kappa_B$ of $\kappa$, $\{E_a\}$ is a local orthonormal basic frame in $Q$ and $\{\theta_a\}$ its $g_Q$-dual 1-form.

The basic Laplacian acting on $\Omega^*_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_Bd_B.$$
If $\mathcal{F}$ is the foliation by points of $M$, the basic Laplacian is the ordinary Laplacian.

3. Transversal twistor spinors

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$. Let $S(\mathcal{F})$ be a foliated spinor bundle of $\mathcal{F}$ and $\langle \cdot, \cdot \rangle$ a hermitian scalar product on $S(\mathcal{F})$. It is well known ([9,13]) that the curvature transform $R^S$ is given as

$$ R^S(X,Y)\Psi = \frac{1}{4} \sum_{a,b} g_Q(R^F(X,Y)E_a, E_b)E_a \cdot E_b \cdot \Psi \text{ for } X,Y \in TM. $$

Moreover, we have [9] that for any vector field $X$

1. $\sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Psi = \frac{1}{4} \sigma \nabla(\pi(X)) \cdot \Psi.$

Let $m : Q \otimes S(F) \to S(F)$ be the Clifford multiplication. Then Ker $m$ is a subbundle of $Q \otimes S(F)$ and there exists a projection $p : Q \otimes S(F) \to$ Ker $m$ onto Ker $m$ given by the formula

$$ p(X \otimes \Psi) = X \otimes \Psi + \frac{1}{q} \sum_{a=1}^{q} E_a \otimes E_a \cdot X \cdot \Psi, $$

where $X \cdot \Psi$ denotes the Clifford multiplication of the vector $X \in Q$ by $\Psi$. There are two operators on the foliated spinor bundle $\Gamma(S(\mathcal{F}))$, the transversal Dirac operator $D'_{tr}$ and the transversal twistor operator $P'_{tr}$. The transversal Dirac operator $D'_{tr}$ and the transversal twistor operator $P'_{tr}$ of $\mathcal{F}$ are defined by

$$ D'_{tr} = m \circ \hat{\pi} \circ \nabla^S, \quad P'_{tr} = p \circ \hat{\pi} \circ \nabla^S, $$

respectively, where $\hat{\pi} : \Gamma(T^*M \otimes S(\mathcal{F})) \to \Gamma(Q^* \otimes S(\mathcal{F})) \cong \Gamma(Q \otimes S(\mathcal{F}))$ is the projection and $\nabla^S$ is a spinor derivation on $S(\mathcal{F})$ induced by (2.2). If it does not cause any confusion, we will henceforward use $\nabla = \nabla^S$. Locally, they are given by respectively

$$ D'_{tr} \Psi = \sum_a E_a \cdot \nabla_{E_a} \Psi, \quad P'_{tr} \Psi = \sum_a E_a \otimes P'_{E_a} \Psi, $$

where $P'_{X} \Psi = \nabla_X \Psi + \frac{1}{q} \pi(X) \cdot D'_{tr} \Psi$ for any $X \in TM$. It was shown ([3],[6]) that the formal adjoint $D'^*_{tr}$ is given by $D'^*_{tr} = D'_{tr} - \kappa$ and that therefore

$$ D_{tr} = D'_{tr} - \frac{1}{2} \kappa, $$
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is a symmetric, transversally elliptic differential operator. On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta \kappa = 0$, it is well-known([2,6,9]) that

\[(3.7)\]
\[D_{tr}^2 \Psi = \nabla_{tr}^{\star} \nabla_{tr} \Psi + \frac{1}{4} K_{\sigma}^{\Psi},\]

where $K_{\sigma}^{\Psi} = \sigma^{\star} + |\kappa|^2$ and

\[(3.8)\]
\[\nabla_{tr}^{\star} \nabla_{tr} \Psi = -\sum_{a} \nabla_{E_a,E_a}^{2} \Psi + \nabla_{\kappa} \Psi.\]

By direct calculation, we have

\[(3.9)\]
\[D_{tr}^2 \Psi = D_{tr}^2 \Psi - \frac{1}{2} \{ \kappa \cdot D_{tr} \Psi + D_{tr} \Psi, \kappa \} - \frac{1}{4} |\kappa|^2 \Psi.\]

Moreover, on an isoparametric transverse spin foliation $\mathcal{F}$, we have

\[(3.10)\]
\[D_{tr} \Psi + \kappa \cdot D_{tr} \Psi = -|\kappa|^2 \Psi - 2 \nabla_{\kappa} \Psi.\]

From (3.9) and (3.10), we have the following proposition.

**Proposition 3.1.** On an isoparametric transverse spin foliation $\mathcal{F}$, we have

\[(3.11)\]
\[D_{tr}^2 \Psi = D_{tr}^2 \Psi + \frac{1}{4} |\kappa|^2 \Psi + \nabla_{\kappa} \Psi.\]

Similarly, we put

\[(3.12)\]
\[P_{tr} \Psi = \sum_{a} E_a \otimes P_{E_a} \Psi,\]

where $P_{X} \Psi = \nabla_{X} \Psi + \frac{1}{q} \pi(X) \cdot D_{tr} \Psi$. Trivially, we have that for any vector $X$

\[(3.13)\]
\[P_{X} \Psi = P_{X}^{\prime} \Psi - \frac{1}{2q} \pi(X) \cdot \kappa \cdot \Psi.\]

We define the subspace $\Gamma_B S(\mathcal{F})$ of basic sections of $S(\mathcal{F})$ by

\[(3.14)\]
\[\Gamma_B S(\mathcal{F}) = \{ \Psi \in \Gamma S(\mathcal{F}) | \nabla_{X} \Psi = 0 \text{ for } X \in \Gamma L \}.\]

Then $D_b = D_{tr} |_{\Gamma_B S(\mathcal{F})}$ preserves the basic sections if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$. This operator $D_b$ is called the **basic Dirac operator** on (smooth) basic sections. A spinor field of kernel of $P_{tr}$ (or kernel of $P_{tr}^{\prime}$) is called the **transversal twistor** (or **W-twistor**) spinor, which satisfies the so-called **transversal twistor** (or **W-twistor**) equation

\[(3.15)\]
\[\nabla_{X} \Psi + \frac{1}{q} \pi(X) \cdot D_{tr} \Psi = 0 \quad \text{or} \quad \nabla_{X} \Psi + \frac{1}{q} \pi(X) \cdot D_{tr} \Psi = 0 \quad X \in TM.\]

In particular, $\Psi \in \text{Ker} P_{tr} \cap \Gamma_B S(\mathcal{F})$ (or $\text{Ker} P_{tr}^{\prime} \cap \Gamma_B S(\mathcal{F})$) is called the **basic twistor** (or **W-basic twistor**) spinor.
Theorem 3.2. If $M$ admits a basic twistor spinor field $\Psi \neq 0$, then $\mathcal{F}$ is minimal.

Proof. Let $\Psi \in \text{Ker}P_{tr}$. Then we have
\begin{equation*}
0 = \sum_a E_a \cdot P_{E_a} \Psi = \sum_a E_a \cdot \nabla_{E_a} \Psi + \frac{1}{q} \sum_a E_a \cdot E_a \cdot D_{\Psi}
\end{equation*}
\begin{equation*}
= D_{\Psi} + \frac{1}{2} \kappa \cdot \Psi - D_{\Psi} = \frac{1}{2} \kappa \cdot \Psi.
\end{equation*}
That is, $\kappa \cdot \Psi = 0$. This implies that $\kappa = 0$. So, $\mathcal{F}$ is minimal. \( \square \)

Remark. From Theorem 3.1, we know that there does not exist a solution of the transversal twistor equation (3.10) if $\mathcal{F}$ is not minimal. So we use the operator $P'_{tr}$ for much information of the foliation $\mathcal{F}$ when it is not minimal. From (3.8) and Theorem 3.1, any basic twistor spinor is a basic W-twistor spinor. But the converse is not true in general.

Theorem 3.3. Let $\Psi \in \text{Ker}P'_{tr}$ be a basic W-twistor spinor. Then for all vector fields $X, Y \in TM$, we have
\begin{equation}
\pi(X) \cdot \nabla_Y \Psi + \pi(Y) \cdot \nabla_X \Psi = \frac{2}{q} g_Q(\pi(X), \pi(Y)) D'_{tr} \Psi.
\end{equation}
Also, the converse holds.

Proof. Assume $P'_{tr} \Psi = 0$, which is equivalent to the transversal W-twistor equation
\begin{equation*}
\nabla_X \Psi + \frac{1}{q} \pi(X) \cdot D'_{tr} \Psi = 0 \quad \text{for any } X \in TM.
\end{equation*}

Multiplying the above equation by a vector field on $M$ we have
\begin{equation*}
\pi(Y) \cdot \nabla_X \Psi + \frac{1}{q} \pi(Y) \cdot \pi(X) \cdot D'_{tr} \Psi = 0,
\end{equation*}
\begin{equation*}
\pi(X) \cdot \nabla_Y \Psi + \frac{1}{q} \pi(X) \cdot \pi(Y) \cdot D'_{tr} \Psi = 0.
\end{equation*}

By summing, we have
\begin{equation*}
\pi(X) \cdot \nabla_Y \Psi + \pi(Y) \cdot \nabla_X \Psi = \frac{2}{q} g_Q(\pi(X), \pi(Y)) D'_{tr} \Psi.
\end{equation*}

Conversely, let (3.11) be valid. Then we have
\begin{equation*}
\sum_a E_a \cdot \pi(X) \cdot \nabla_{E_a} \Psi + \sum_a E_a \cdot E_a \cdot \nabla_X \Psi = \frac{2}{q} \sum_a g_Q(\pi(X), E_a) E_a \cdot D'_{tr} \Psi.
\end{equation*}

Hence we have
\begin{equation*}
\nabla_X \Psi + \frac{1}{q} \pi(X) \cdot D'_{tr} \Psi = 0.
\end{equation*}

This implies that $\Psi$ is a basic W-twistor spinor. \( \square \)
Theorem 3.4. Let $\Psi \in \text{Ker} P_t'$ be a basic W-twistor spinor. Then we have

\begin{align}
D_t'^2 \Psi &= \frac{q}{4(q-1)} \sigma \nabla \Psi \\
\nabla_X D_t'^2 \Psi &= \frac{q}{2(q-2)} \left( \frac{\sigma \nabla}{2(q-1)} \pi(X) - \rho \nabla (\pi(X)) \right) \Psi.
\end{align}

Proof. Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all $a$. From (3.10), we have at $x$ that for any basic W-twistor spinor $\Psi$

\begin{equation}
\sum_a \nabla E_a \nabla E_a \Psi + \frac{1}{q} D_t'^2 \Psi = 0.
\end{equation}

On the other hand, by direct calculation, we have

\begin{equation}
D_t'^2 \Psi = -\sum_a \nabla E_a \nabla E_a \Psi + \frac{1}{q} \sigma \nabla \Psi.
\end{equation}

From (3.14) and (3.15), the first equation (3.12) is proved.

Next, let $X$ be a local vector field arising from a vector in $T_x M$ by parallel displacement along transversal geodesics. From (3.10), we have that at $x$,

\begin{align}
\nabla E_a \nabla_X \Psi + \frac{1}{q} \pi(X) \cdot \nabla E_a D_t' \Psi = 0, \\
\nabla_X \nabla E_a \Psi + \frac{1}{q} E_a \cdot \nabla_X D_t' \Psi = 0.
\end{align}

Hence we have

\begin{equation}
R^S(X, E_a) \Psi = \frac{1}{q} \{ \pi(X) \cdot \nabla E_a D_t' \Psi - E_a \cdot \nabla_X D_t' \Psi \}.
\end{equation}

Since $i(X) R^S = 0$ for $X \in \Gamma L$, if $\Psi \in \Gamma_B S(F)$, then $D_t' \Psi \in \Gamma_B S(F)$. From (3.3) and (3.16), we have

\begin{align}
\rho \nabla (\pi(X)) \cdot \Psi &= -2 \sum_a E_a \cdot R^S(X, E_a) \Psi \\
&= -\frac{2}{q} \sum_a E_a \cdot \{ \pi(X) \cdot \nabla E_a D_t' \Psi - E_a \cdot \nabla_X D_t' \Psi \} \\
&= -\frac{2}{q} \{ (q-2) \nabla_X D_t' \Psi - \pi(X) \cdot D_t'^2 \Psi \}.
\end{align}

From (3.13), we obtain the second equation. $\square$

Now we prove a further condition for $\Psi$ being a transversal W-twistor spinor. Let us define the bundle map $K : TM \to Q$ by

\begin{equation}
K(X) = \frac{1}{q-2} \{ \frac{\sigma \nabla}{2(q-1)} \pi(X) - \rho \nabla (\pi(X)) \}.
\end{equation}
From (3.13), it is trivial that for any transversal W-twistor spinor $\Psi$

$$(3.23)\quad \nabla_X D'_{tr} \Psi = \frac{q}{2} K(X) \cdot \Psi.$$ 

We consider the bundle $E = S(\mathcal{F}) \oplus S(\mathcal{F})$ and the covariant derivative $\nabla^E$ in $E$ defined by

$$(3.24)\quad \nabla^E_X \left( \begin{array}{c} \Phi \\ \Psi \end{array} \right) = \left( \begin{array}{c} \nabla_X \Phi + \frac{1}{q} \pi(X) \cdot \Psi \\ \nabla_X \Psi - \frac{q}{2} K(X) \cdot \Phi \end{array} \right).$$

Then we have the following Theorem.

**Theorem 3.5.** For any basic W-twistor spinor $\Phi \in \Gamma_B S(\mathcal{F})$, it holds

$$\nabla^E \left( \begin{array}{c} \Phi \\ D'_{tr} \Phi \end{array} \right) \equiv 0.$$ 

Conversely, if $\left( \begin{array}{c} \Phi \\ \Psi \end{array} \right) \in \Gamma_B E$ is $\nabla^E$-parallel, then $\Phi$ is a basic W-twistor spinor and $\Psi = D'_{tr} \Phi$.

**Proof.** Let $\Phi \in \Gamma_B S(\mathcal{F})$ be a basic W-twistor spinor. From (3.19), we have

$$\nabla^E_X \left( \begin{array}{c} \Phi \\ D'_{tr} \Phi \end{array} \right) = \left( \begin{array}{c} \nabla_X \Phi + \frac{1}{q} \pi(X) \cdot D'_{tr} \Phi \\ \nabla_X D'_{tr} \Phi - \frac{q}{2} K(X) \cdot \Phi \end{array} \right).$$

From (3.10) and (3.18), we have

$$\nabla^E \left( \begin{array}{c} \Phi \\ D'_{tr} \Phi \end{array} \right) \equiv 0.$$ 

Conversely, let $\left( \begin{array}{c} \Phi \\ \Psi \end{array} \right) \in \Gamma_B E$ be a $\nabla^E$-parallel section:

$$\nabla^E \left( \begin{array}{c} \Phi \\ \Psi \end{array} \right) = 0.$$ 

Then by definition of $\nabla^E$, we have

$$\nabla_X \Phi + \frac{1}{q} \pi(X) \cdot \Psi = 0 \quad \text{for any } X \in TM$$

and then

$$\sum_a E_a \cdot \nabla_{E_a} \Phi + \sum_a \frac{1}{q} E_a \cdot E_a \cdot \Psi = 0,$$
where \( \{ E_a \} \) is an orthonormal base frame of \( Q \). Hence \( D'_t \Phi = \Psi \). This implies that \( \Phi \) is a solution of the transversal W-twistor equation. \( \square \)

4. Transversal Killing spinor

For a basic function \( f \), the spinor field \( \Psi \in \Gamma S(F) \) satisfies the transversal Killing equation if

\[
\nabla^f_X \Psi \equiv \nabla_X \Psi + f(x) \psi_X \Psi = 0 \quad \text{for any } X \in TM.
\]

In this case, \( \Psi \) is called a transversal Killing spinor on \( F \).

Lemma 4.1. If \( \Psi \) is a transversal Killing spinor, then the associate vector field \( X_\Psi \) defined by

\[
X_\Psi = i \sum_{a=1}^q \langle \Psi, E_a \cdot \Psi \rangle E_a
\]

is a transversal Killing vector field, i.e., \( \theta(X_\Psi)g_Q = 0 \).

Proof. Generally, we have that for any \( Y, Z \in \Gamma Q \)

\[
(\theta(X_\Psi)g_Q)(Y, Z) = g_Q(\nabla_Y \pi(X), Z) + g_Q(Y, \nabla_Z \pi(X)).
\]

Let \( \nabla^f_X \Psi = 0 \). Equivalently, \( \nabla_X \Psi = -f(x) \psi_X \Psi \). Hence we have

\[
\nabla_Y X_\Psi = i \sum_{a=1}^q Y \langle \Psi, E_a \cdot \Psi \rangle E_a
\]

\[
= i \sum_{a=1}^q \{ \langle \nabla_Y \Psi, E_a \cdot \Psi \rangle + \langle \Psi, E_a \cdot \nabla_Y \Psi \rangle \} E_a
\]

\[
= -if \sum_{a=1}^q \{ \langle Y \cdot \Psi, E_a \cdot \Psi \rangle + \langle \Psi, E_a \cdot Y \cdot \Psi \rangle \} E_a.
\]

Hence we have

\[
g_Q(\nabla_Y X_\Psi, Z) = -if \{ \langle Y \cdot \Psi, Z \cdot \Psi \rangle + \langle \Psi, Z \cdot Y \cdot \Psi \rangle \}.
\]

Similarly,

\[
g_Q(Y, \nabla_Z X_\Psi) = -if \{ \langle Z \cdot \Psi, Y \cdot \Psi \rangle + \langle \Psi, Y \cdot Z \cdot \Psi \rangle \}.
\]

Hence we have

\[
(\theta(X_\Psi)g_Q)(Y, Z) = g_Q(\nabla_Y X_\Psi, Z) + g_Q(Y, \nabla_Z X_\Psi) = 0. \quad \square
\]
Lemma 4.2. If $\Psi$ is the transversal Killing spinor, then $|\Psi|^2$ is constant.

Proof. Let $\Psi$ is the transversal Killing spinor, i.e., $\nabla_X \Psi = -f \pi(X) \cdot \Psi$. For any $X \in TM$

$$X|\Psi|^2 = \langle \nabla_X \Psi, \Psi \rangle + \langle \Psi, \nabla_X \Psi \rangle = -f \{ \langle \pi(X) \cdot \Psi, \Psi \rangle + \langle \Psi, \pi(X) \cdot \Psi \rangle \} = 0.$$ 

So $|\Psi|^2$ is constant. $\square$

Proposition 4.3. If $M$ admits a transversal Killing spinor $\Psi$ with $\nabla_X^f \Psi = 0$, then

1. $f$ is constant and $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$
2. $\mathcal{F}$ is transversally Einsteinian with constant transversal scalar curvature $\sigma^\nabla$.

Proof. By direct calculation, we have

$$\sum_a E_a \cdot R^a_{XE_a} \Psi = -\frac{1}{2} \rho^\nabla(X) \cdot \Psi + 2(q-1)f^2 X \cdot \Psi - qX(f) \Psi - grad^\nabla(f) \cdot X \cdot \Psi$$

for $X \in \Gamma Q$. Since $\nabla^f \Psi = 0$, we have

$$0 = -\frac{1}{2} \rho^\nabla(X) \cdot \Psi + 2(q-1)f^2 X \cdot \Psi - qX(f) \Psi - grad^\nabla(f) \cdot X \cdot \Psi.$$ 

If we put $X = grad^\nabla(f)$, then

$$< -\frac{1}{2} \rho^\nabla(X) \cdot \Psi + 2(q-1)f^2 X \cdot \Psi, \Psi > = (q-1)|grad^\nabla(f)|^2 |\Psi|^2.$$ 

Since the left hand side is pure imaginary and right hand side is real, we have

$$|grad^\nabla(f)| = 0.$$ 

Since $f$ is a basic function, $f$ is constant. Hence from (4.2) we have

$$-\frac{1}{2} \rho^\nabla(X) \cdot \Psi + 2(q-1)f^2 X \cdot \Psi = 0.$$ 

Hence we have

$$\rho^\nabla(X) = 4(q-1)f^2 X.$$ 

This implies that $\mathcal{F}$ is transversally Einsteinian. From (2.3), we have $\sigma^\nabla = 4q(q-1)f^2$. $\square$

Theorem 4.4. If $\Psi$ is a transversal Killing spinor, then

$$|D_{tr}\Psi|^2 = \frac{1}{4} \left( \frac{q}{q-1} \sigma^\nabla + |\kappa|^2 \right) |\Psi|^2$$ 

(4.4)

$$\Re < D_{tr}\Psi, \kappa \cdot \Psi > = -\frac{1}{2} |\kappa|^2 |\Psi|^2.$$ 

(4.5)
Proof. Let $\Psi$ be the transversal Killing spinor. From Proposition 4.3, we have

$$\nabla_X \Psi = -f X \cdot \Psi, \quad D_{tr} \Psi = f q \Psi - \frac{1}{2} \kappa \cdot \Psi,$$

where $f^2 = \frac{q \sigma}{4 q q - 1}$.

From the second equation in (4.6), we get

$$< D_{tr} \Psi, D_{tr} \Psi > = < f q \Psi - \frac{1}{2} \kappa \cdot \Psi, f q \Psi - \frac{1}{2} \kappa \cdot \Psi > = (f^2 q^2 + \frac{1}{4} |\kappa|^2) < \Psi, \Psi >.$$

Hence we have

$$|D_{tr} \Psi|^2 = \frac{1}{4} \left( \frac{q}{q - 1} \sigma \nabla + |\kappa|^2 \right) |\Psi|^2.$$

The equation (4.5) is trivial from (4.6). \( \square \)

**Corollary 4.5.** If there exists an eigenspinor $\Psi$ of $D_b$ with $\nabla f \Psi = 0$, then $F$ is minimal.

**Theorem 4.6.** On the minimal foliation $F$, every transversal Killing spinor is an eigenspinor.

Proof. Let $\Psi$ be the transversal Killing spinor. From (4.6), if $F$ is minimal, then

$$D_b \Psi = f q \Psi.$$

This implies that $\Psi$ is an eigenspinor. \( \square \)

**5. Eigenvalue estimates**

By a straightforward calculation, we have that for any spinor field $\Psi \in \Gamma S(F)$

$$|P'_{tr} \Psi|^2 = |\nabla_{tr} \Psi|^2 - \frac{1}{q} |D'_{tr} \Psi|^2.$$

From (3.7), we have that

$$\int_M |P'_{tr} \Psi|^2 = \int_M |D_{tr} \Psi|^2 - \frac{1}{4} K^\sigma |\Psi|^2 - \frac{1}{q} |D'_{tr} \Psi|^2.$$

Since $D_{tr} \Psi = D'_{tr} \Psi - \frac{1}{2} \kappa \cdot \Psi$, we have

$$|D_{tr} \Psi|^2 = |D'_{tr} \Psi|^2 - \frac{1}{4} |\kappa|^2 |\Psi|^2 - Re < D_{tr} \Psi, \kappa \cdot \Psi >.$$
From (5.2) and (5.3), we have
\[
\int_M |P'_\tau \psi|^2 = \frac{q-1}{q} \int_M \{ |D_{\tau} \psi|^2 - \frac{q}{4(q-1)} (K^\sigma + \frac{1}{q} |\kappa|^2) \} |\psi|^2 \\
+ \int_M \text{Re} < D_{\tau} \psi, \kappa \cdot \psi >.
\]

Let \( D_b \psi = \lambda \psi \). Since \( < \psi, \kappa \cdot \psi > \) is pure imaginary, we have
\[
\int_M |P'_\tau \psi|^2 = \frac{q-1}{q} \int_M \{ \lambda^2 - \frac{q}{4(q-1)} (K^\sigma + \frac{1}{q} |\kappa|^2) \} |\psi|^2.
\]

Hence we have the following theorem.

**Theorem 5.1.** Let \((M, g_M, \mathcal{F})\) be a compact Riemannian manifold with a transverse spin foliation \( \mathcal{F} \) of codimension \( q \geq 2 \) and bundle-like metric \( \tilde{g}_M \). Then any eigenvalue \( \lambda \) of the basic Dirac operator \( D_b \) satisfies
\[
\lambda^2 \geq \frac{q}{4(q-1)} \inf_M (K^\sigma + \frac{1}{q} |\kappa|^2).
\]

In the limiting case, \( \mathcal{F} \) is minimal, transversally Einsteianian with constant transversal scalar curvature \( \sigma^\nabla \).

**Proof.** Let us consider \( D_b \psi = \lambda \psi \) with \( \lambda^2 = \frac{q}{4(q-1)} \inf_M (K^\sigma + \frac{1}{q} |\kappa|^2) \). Then \( P'_\tau \psi = 0 \). Hence from (3.11) and (3.17), it is trivial that
\[
\sigma^\nabla = \text{constant}, \quad |\kappa| = 0.
\]

That is, \( \mathcal{F} \) is minimal. Moreover, from (3.18), the foliation \( \mathcal{F} \) is transversally Einsteianian. \( \square \)

**References**


